Complementary Collections in Ligeti's *Désordre*

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Abstract. This paper examines one aspect of Ligeti's approach to writing music that is neither tonal nor atonal—the use of complementary collections to achieve what Richard Steinitz has termed combinatorial tonality. After a brief introduction, the paper explores properties of the intervallic content both within and between complementary collections, which I term the intra- and inter-harmonies. In particular, the inter-harmonies are useful in understanding harmonic control in works based on complementary collections, as demonstrated by revisiting Lawrence Quinnett's analysis of Ligeti's first Piano Étude, *Désordre*.

Keywords: Ligeti, *Désordre*, complementary collections, combinatorial tonality

1 Introduction

Figure 1 shows the opening of Ligeti's first Piano Étude, *Désordre*. A remarkable feature of this étude is that the right hand plays only the white keys while the left hand plays only the black keys. The étude thus systematically divides the aggregate into two quite familiar complementary collections—diatonic and pentatonic. Due to the overlapping registers (proximity) and similarity of contour in the ascending scalar fragments (common fate), it can be difficulty to separate the two hands into independent psychological streams, thus making the diatonic and pentatonic collections difficult to hear separately. (See Bregman [3] for the importance of pitch proximity and common fate in the formation of independent auditory streams.) Instead, it is much easier to hear the *between* hand note-against-note harmonic intervals, which we might term the *inter-harmonies*.

Fig. 1: Opening of *Désordre*

As the étude progresses, the hands gradually drift apart, the accents in the two hands become desynchronized, and the durations between accents in each hand are gradually shortened, leading to a fragmentation of the scalar segments. Near the climax of the étude (see Figure 6*b*), both the lack of pitch proximity and common fate in the melodic contours strongly encourages the formation of two independent streams, making it difficult to hear the harmonic relations between the hands, but relatively easy to hear the *within* stream intervallic relationships, which we might term the *intra-harmonies*. From the opening to the climax of *Désordre* there is thus a change in focus from the inter-harmonies to the intra-harmonies. Richard Steinitz [12] has referred to this interplay of complementary collections as "combinatorial tonality."

While there are clear precedents in the use of complementary collections, Ligeti's extensive use of opposed collections in his late works and, in particular, his exploration of the unfamiliar harmonic possibilities between collections and the ways in which a listener's attention can be focused on the relations within (inter) and between (intra) collections is something very different. Indeed, the technique of complementary collections may represent Ligeti's most systematic approach to achieving his goal of creating music that is neither tonal not atonal in his late works. (See Ligeti's own comments in [2].) As such, our lack of understanding of the harmonic relations between complementary collections takes on greater significance. The current paper explores this aspect of Ligeti's combinatorial tonality, focusing on the relevant mathematical properties of complementary collections and the first part of *Désordre*, as a preliminary step to a greater understanding of Ligeti's exploration and realizations of the theoretical properties and corresponding possibilities of complementary collections.[1]

2 Properties of intra- and inter-harmonies

We begin examining the harmonic relationship between complementary collections by looking at the interval content (IC) from one pitch-class set to another using Lewin's [8] interval function (IFUNC):

$$\mathrm{IC}_{A,B}(k) = \mathrm{IFUNC}(A,B)(k) = |\{(a,b) \in A \times B, b-a=k\}|.^2$$

The interval function is a histogram of the pc intervals by which a member of one set can move to a member of the other set, yielding a 12-valued vector of pc interval multiplicities. For example, the interval content from $\{\mathrm{C},\mathrm{D}\}$ to $\{\mathrm{C}\sharp,\mathrm{D}\sharp\}$ is $(0,2,0,1,0,0,0,0,0,0,0,1)$, indicating that there are two ways to move by pc interval 1 (C→C♯ and D→D♯), one way to move by pc interval 3 (C→D♯), one way to move by pc interval 11 (D→C♯), and no ways to move by any other interval.[3] In the special instance of the interval content from a set to itself, the interval content *within* a set, we will use the shorthand $\mathrm{IC}_{A,A} = \mathrm{IC}_A$.

1 My thanks to Nancy Rogers and an anonymous reviewer for comments that greatly improved this paper.

2 The reader is strongly directed to Amiot [1] for an excellent and detailed presentation of the interval function and its relation to recent applications of the discrete Fourier transform in music theory.

3 Multiplicity of pc interval i is indicated by the i^{th} component of the interval function, which begins with pc interval 0.

For complementary collections A and $\bar{A}$, the *intra-harmonies* are a combination of the interval content within each collection separately:

$$A_{\text{intra}} = \text{IC}_A + \text{IC}_{\bar{A}}.$$

For example, setting W to be the white-key diatonic collection ($W = \{0, 2, 4, 5, 7, 9, 11\}$), the intra-harmonies for the white-key/black-key complementary collections are given by

$$\begin{aligned} W_{\text{intra}} &= \text{IC}_W + \text{IC}_{\bar{W}} \\ &= (7, 2, 5, 4, 3, 6, 2, 6, 3, 4, 5, 2) + (5, 0, 3, 2, 1, 4, 0, 4, 1, 2, 3, 0) \\ &= (12, 2, 8, 6, 4, 10, 2, 10, 4, 6, 8, 2). \end{aligned}$$

Similarly, for complementary collections, the *inter-harmonies* combine the interval content that obtains exclusively *between* the two collections:

$$A_{\text{inter}} = \text{IC}_{A,\bar{A}} + \text{IC}_{\bar{A},A}.$$

For complementary collections, A and $\bar{A}$, $\text{IC}_{A,\bar{A}} = \text{IC}_{\bar{A},A}$. Thus,

$$A_{\text{inter}} = 2 \cdot \text{IC}_{A,\bar{A}}.$$

For example, again setting W to be the white-key diatonic collection, the inter-harmonies for the white-key/black-key complementary collections are given by

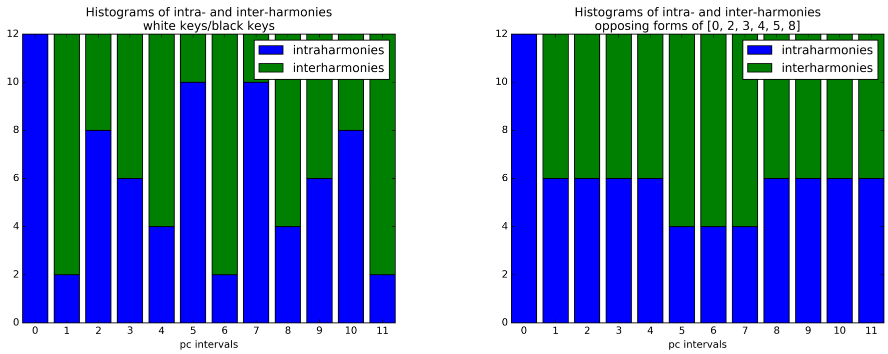
$$\begin{aligned} W_{\text{inter}} &= 2 \cdot \text{IC}_{W,\bar{W}} \\ &= 2 \cdot (0, 5, 2, 3, 4, 1, 5, 1, 4, 3, 2, 5) \\ &= (0, 10, 4, 6, 8, 2, 10, 2, 8, 6, 4, 10). \end{aligned}$$

For a given pair of complementary collections, any pc interval must occur either within or between the collections, and thus the intra- and inter-harmonies exhaust the set of possible pc intervals:

$$A_{\text{intra}} + A_{\text{inter}} = \text{IC}_{\mathbb{Z}_n} = (n, \ldots, n).$$

Figure 2 shows graphs of the intra- and inter-harmonies for two different pairs of complementary collections. Note that the distributions of intra- and inter-harmonies for Figure 2a are fairly uneven, while the distribution in b is nearly flat. Collections in which the interval content is highly uneven may be thought to be more distinctive, since the interval content is dominated by only a few (and therefore salient) pc intervals. Contrarily, when the interval content is very flat, the corresponding collections cannot be typified by a limited number of distinct pc intervals. In this sense, there is a strong correlation between the "distinctiveness" of a collection and the unevenness of its interval content.

We can measure the unevenness of a collection's interval content by taking its standard deviation. For example, the interval content of the "flat" hexachord from Figure 2b has a



(a) (b)

Fig. 2: Histograms of intra- and inter-harmonies for a) white key/black key collections and b) "flat" hexachord {C, D, D♯, E, F, G♯} and its complement

standard deviation of $\sigma(\mathrm{IC}_{\{0,2,3,4,5,8\}}) = 1.0$, while that of the whole-tone collection (with its maximally uneven interval content of all even intervals and no odd intervals) has a standard deviation of $\sigma(\mathrm{IC}_{\{0,2,4,6,8,10\}}) = 3.0$. The standard deviation of the white-key interval content lies between these two extremes: $\sigma(\mathrm{IC}_{\{0,2,4,5,7,9,11\}}) \approx 1.66$.[4]

As measured by the standard deviation of interval content, a collection and its complement are equally distinctive:[5]

$$\sigma(\mathrm{IC}_A) = \sigma(\mathrm{IC}_{\bar{A}}).$$

Moreover, the distinctiveness of the intra-harmonies is the same as the inter-harmonies:

$$\sigma(A_{\mathrm{intra}}) = \sigma(A_{\mathrm{inter}}).$$

Since Ligeti's favorite complementary collections, including diatonic, pentatonic, whole-tone, Guidonian and similar collections, are all highly distinctive, this ensures that the interval content between these collections and their complements will also be highly distinctive.

This does not, however, guarantee that the intra- and inter-harmonies will also be highly differentiated. In order to measure this differentiation, we can take the magnitude of the difference of the two vectors:

$$\|A_{\mathrm{inter}} - A_{\mathrm{intra}}\|_2.$$

For example, let W be the white-key collection and X be the "flat" hexachord {0,2,3,4,5,8}. The Euclidean distance between the intra- and inter-harmonies for the diatonic collection is nearly 24 ($\|W_{\mathrm{inter}} - W_{\mathrm{intra}}\|_2 \approx 23.98$), while the corresponding distance for the flat hexachord is nearly only 14 ($\|X_{\mathrm{inter}} - X_{\mathrm{intra}}\|_2 \approx 13.86$), reflecting the high differentiation

[4] The distinctiveness of a collection, A, can also be measured in terms of the magnitude of its interval content, $\|\mathrm{IC}_A\|_2$. (See Callender [4].) For the present purposes, the standard deviation is preferable.

[5] This follows directly from the complement theorem. See Hanson [6] and Lewin [7].

of intra- and inter-harmony distributions in Figure $2a$ and relatively low differentiation in $2b$.[6]

More generally for complementary collections, there is a strong relationship between the differentiation of intra- and inter-harmonies and the distinctiveness of the collections. If complementary collections A and $\bar{A}$ are the same cardinality, then

$$\|A_{\text{inter}} - A_{\text{intra}}\|_2 \propto \sigma(IC_A).$$

(Specifically, for a chromatic universe of C pitch classes, $\|A_{\text{inter}} - A_{\text{intra}}\|_2 = 4\sqrt{C} \cdot \sigma(IC_A)$.) If the cardinalities are nearly equal, then the relation is nearly, though not exactly, proportional. Thus, highly distinctive collections indeed posses highly differentiated intra- and inter-harmonies. By working with highly distinctive collections, Ligeti ensures that there will be maximal variation between the melodic and harmonic components of the resulting combinatorial tonality.

3 *Désordre*

Returning to the opening of *Désordre* (Figure 1), we would like to answer the following question: To what extent does Ligeti exert control over the note-against-note harmonies in the étude? The opening of *Désordre* consists of two layers that persist throughout the entire étude. The accented notes correspond to a highly complex isorhythmic structure, detailed in Kinzler [9], in which the left and right hands have very similar but non-identical *colores* (sequences of pitches) and *taleae* (sequences of durations). While the accents of the two hands are synchronized in the beginning of the étude, they quickly become misaligned, due to the slight difference in their *taleae*. Unaccented notes are not a part of the isorhythmic structure, but rather form a second layer consisting of generally ascending scalar fragments used to smoothly connect the accented notes. Perhaps the harmonic relations at any point are simply the result of the particular and temporary configuration of the two hands within the overarching isorhythmic structure. If this is the case, then over a large enough span of time the observed harmonies will be equivalent to the result of repeated random selection from the distribution of possible harmonies between the two collections. In other words, as the étude progresses, the distribution of observed harmonies will converge on the distribution of the inter-harmonies.

In the opening line of *Désordre* (Figure 1), tritones and minor sixths predominate, while there are almost no minor second/major sevenths or perfect fourths/fifths. The lack of interval class (ic) 5 is easily explained by the relative lack of this interval in the inter-harmonies. However, the relative lack of ic 1 may indicate some degree of control on the part of the composer, since there are plenty of minor seconds/major sevenths spanning the

[6] Measuring the distance between intra- and inter-harmonies using other metrics, such as angular (or cosine) distance, yields similar relative distances. (See Rogers [11].) The Euclidean metric is sufficient and advantageous for the present purposes.

two collections. Does this favoring of tritones and minor sixths (major thirds) over minor seconds/major sevenths persist?

Figure 3 gives the normalized actual (observed) and expected interval distributions for the first section of the étude, concluding with the climax on the first downbeat on the sixth page of the published score.[7] (Intervals are reckoned between hands interpreted as pitch-class sets.) There is a noticeable emphasis on ic 6 and de-emphasis on ic 1. In his dissertation on harmony and counterpoint in Ligeti's *Études*, Quinnett [10] presents a comparison of observed and expected interval counts of the first section of *Désordre* divided into two parts, with the second part beginning where the accents of the two hands temporarily become (mostly) realigned (t_2, beginning just before the bottom system on page 2 of the score). (See Figures 4 and 5.) Quinnett notes that the interval profile of the first section heavily favors tritones and minor sixths/major thirds over minor seconds/major sevenths, while the interval content for the second section is much more similar to the expected distribution.

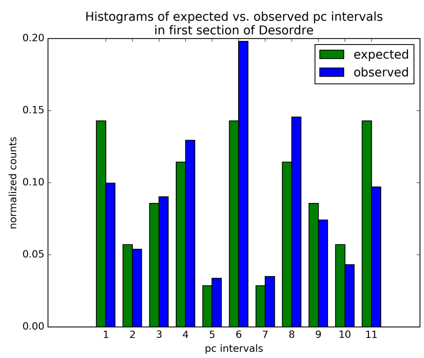
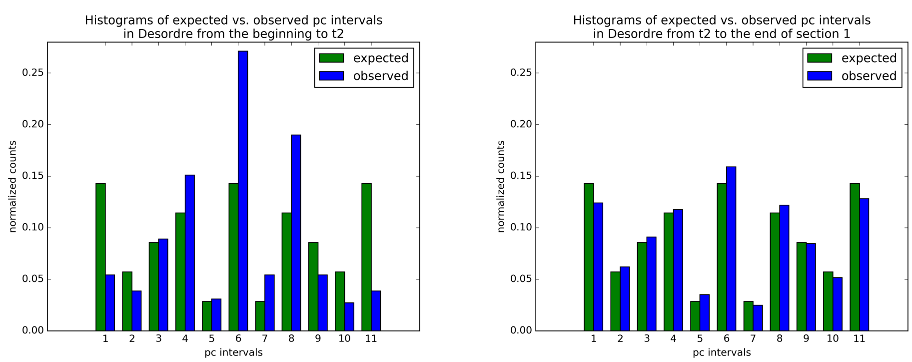


Fig. 3: Histograms of observed and expected intervals in the first section of *Désordre*

Why are the observed and expected distributions of the second part much more similar than in the first section? Quinnett suggests that this is due to the progressive rhythmic diminution of the isorhythmic structure that begins in the second part. As the durations of the talea decrease, the density of accented notes from the color increases, and the freedom that Ligeti had in his choice of pitches diminishes. Thus, as discussed above, we would expect the interval content to become increasingly governed by the distribution of intervals in the inter-harmonies. While the histograms of Figures 3 and 5 and Quinnett's explanations are suggestive, in what follows we will briefly consider the statistical significance and size of the differences between the observed and expected interval distributions, look at the progression of the observed intervals at a finer level of detail, and consider alternative

[7] Statistical analysis of *Désordre* was greatly aided by Cuthbert's music21 [5], which is a Python toolkit for computer-aided musicology.

Fig. 4: Passage before and after the realignment of accents, marked by the vertical bar at t_2. Instances of interval class 1 are marked with asterisks.



(*a*) (*b*)

Fig. 5: Histograms of observed and expected intervals in the *a*) first and *b*) second part of the first section of *Désordre* (after Quinnett [10])

explanations for the convergence of observed and expected intervals toward the end of the first section.[8]

In order to compare the actual distribution of between-hand intervals in *Désordr*e with the expected distribution based on the inter-harmonies, χ^2 goodness of fit tests were conducted for various time spans within the first section. In all cases the null hypothesis is that the observed intervals are consistent with the distribution of the inter-harmonies. The alternative hypothesis—that the observed intervals are not consistent with this distribution—implies that Ligeti is exerting control over the harmonic quality of a given time span in ways that cut against simple scalar connections between notes of the isorhythmic structure. Table 1 gives observed and expected interval counts in the first section of *Désordre* along with the corresponding χ^2 statistic and p-value. The test confirms the intuition that the differences between the two distributions are highly significant, though it does not address the size of this difference (see below).

pc intervals	1	2	3	4	5	6	7	8	9	10	11
observed	74	40	67	96	25	147	26	108	55	32	72
expected	106.0	42.4	63.6	84.8	21.2	106.0	21.2	84.8	63.6	42.4	106.0
std. residuals	**-3.11**	-0.37	0.43	1.22	0.83	**3.98**	1.04	2.52	-1.08	-1.60	**-3.30**

$$\chi^2 = 50.05,\ p < 0.000001$$

Table 1: Contingency table of observed and expected intervals in the first section of *Désordre*, standardized residuals, and corresponding χ^2 statistic and p-value

The standardized residuals ($\frac{O-E}{\sqrt{E}}$, where O and E are the observed and expected counts, respectively) quantify the contribution of each pc interval to the overall χ^2 value. The most significant values in this row (shown in bold) identify the categories that are driving the lack of fit between the two distributions. In particular, observed pc intervals 1 and 11 are significantly less frequent than expected, while pc interval 6 is significantly more frequent than expected, confirming intuitions based on Figures 3 and 5.

In Table 2 the first section of *Désordre* is divided into various subsections, based on five time points measured in eighth notes from the beginning of the étude: $t_0 = 0$ is the beginning of the étude where there are very few instances of interval class 1 (see Figure 1); $t_1 = 160$ marks the beginning of a passage with slightly increasing presence of ic 1 (see Figure 4); $t_2 = 248$ marks the realignment of accents between the two hands, accompanied by a return to very few instances of ic 1 (see Figure 4); at $t_3 = 316$ durations of the isorhythm are progressively shortened and ic 1 becomes much more prevalent (see Figure 6a); and $4 = 634$ is the end of the first section, which includes accents in both hands on every pulse (see Figure 6b). The final column of the table reports the phi coefficient, ϕ,

[8] Comparison of interval counts with the inter-harmonies in Désordre in both Quinnett's treatise and the current paper stem from our conversations while Quinnett was a student at Florida State University.

which is a χ^2-based measure of effect size: $\phi = \sqrt{\frac{\chi^2}{n}}$, where n is the number of samples in the data. Larger values for ϕ indicate a greater difference between the two distributions.[9]

(a)

(b)

Fig. 6: a) section before and after time point t_3, b) section immediately before the end of the first section (t_4). Asterisks indicate instances of interval class 1.

begin	end	p-value	ϕ
t_0	t_4	< 0.0001	0.26
t_0	t_2	< 0.0001	0.60
t_2	t_4	0.91	0.10
t_0	t_1	< 0.0001	0.72
t_1	t_2	0.01	0.49
t_2	t_3	< 0.0001	0.79
t_3	t_4	0.96	0.10

Table 2: Comparison of observed and expected intervals for various time spans in the first section of *Désordre* with p-values and reduced phi coefficients (ϕ_ν) for χ^2 goodness of fit tests

The results of Table 2 suggest that perhaps Ligeti exerted a finer degree of intervallic control than can be captured by dividing the opening section into two large parts as in Figure 5. Row 1 of the table repeats the test from Table 1 of the entirety of the first

[9] Note that because there are more than a single degree of freedom in the data, ϕ is not normalized to a maximum of 1.

section. Rows two and three divide the section into two parts. In first part, from t_0 to t_2, the difference between observed and expected intervals is significant and also has a larger effect size than the entire section. In the second part, from t_2 to t_4, the differences are not significant and the effect size is correspondingly very low. Time point t_1 divides the time span from t_0 to t_2 into two subparts in rows four and five, and time point t_3 similarly divides the span from t_2 to t_4 in rows six and seven. In both pairs of rows the first subpart differs strongly from the expected interval distribution, while the effect size is lower in the second subpart due to the increase in the prevalence of interval class 1. This is particularly true in the time span beginning at t_3. The upshot is that changes in the interval distribution support a division of the opening section into two parts, with a significant return to synchronized accents and avoidance of ic 1 at t_2. This sense of return is enhanced by the slight increase in ic 1 in the span from t_1 to t_2.

These changes in the intervallic distribution over the course of the first section can be seen more clearly in Figure 7, which plots ϕ for a moving window of 65 eighth notes. Here, ϕ is based on a χ^2 goodness of fit test consisting of only two categories of intervals: those that belong to interval class 1 and those that do not. This graph demonstrates that the changes in harmonic content noted in Table 2 happen abruptly rather than gradually. (This is evident even though the transitions between regions of higher and lower values for ϕ are smoothed by the moving-window analysis.) Regions of lower values for ϕ are either mostly or almost entirely below the lines indicating various significance levels, whereas regions of higher values are almost entirely above these lines. These abrupt transitions as well as the return at t_2 of the intervallic content of the opening complicate the earlier explanation of the changing harmonic distribution over the course of the section. If these changes were simply the result of the progressive rhythmic diminution of the isorhythmic structure (beginning at t_3) and the consequent lack of harmonic freedom, the values for ϕ in Figure 7 would remain consistently high until t_3 and then gradually decrease. Ligeti appears to be exerting control in switching from one distribution to the other.

To the extent that the quickening of the isorhythms plays a role in the differing intervallic distributions of the two parts, might there be other factors involved? Perhaps as the hands drift apart toward the registral extremes of the piano, Ligeti became less concerned with note-against-note harmonies, since the increased distance between the two hands encourages the perception of two independent streams and makes it difficult to perceive the quality of the between-hand intervals. The challenge in assessing the relative strengths of these two explanations is that interval size and *talea* durations are strongly (inversely) correlated.

One approach to separating these factors is to divide intervals for each time point by size and whether or not an accent is present. (Recall that accents always and only accompany elements of the isorhythm, so the presence of accents can be used as an indicator for the presence of the isorhythm.) In Figure 8 intervals throughout the first section of *Désordre* are divided into categories based on small (S) or large (L) interval size and presence (T) or lack (F) of accents. (Small intervals are no larger than two octaves. Pitch intervals in-

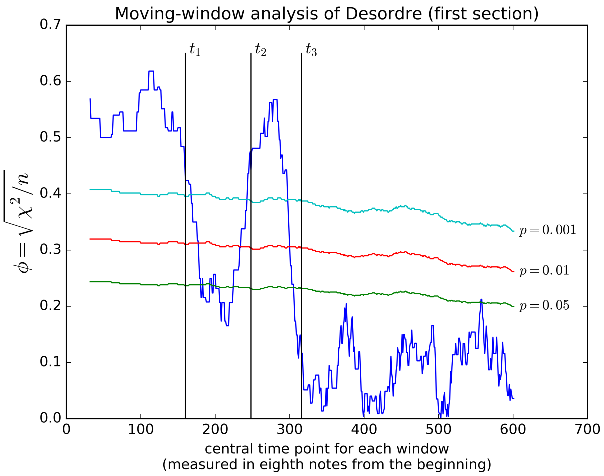


Fig. 7: Moving-window analysis of observed vs. expected frequency of interval class 1 in the first section of *Désordre*. Larger values for ϕ correspond to a greater difference between observed and expected frequencies. Window size is 65 eighth notes. Expected frequencies are based on the inter-harmonies. Time points t_1, t_2, and t_3 are indicated with vertical lines. Lines running across the graph indicate values for ϕ corresponding to various levels of significance.

volving octaves are reckoned from the lower note of the octave.) For example, the interval in the first eighth note of the étude belongs to the category 'S-T', since it is less than two octaves and the time point contains at least one accent. The interval of 30 semitones on the unaccented time point immediately before t_3 (Figure 6a) belongs to the category 'L-F'.

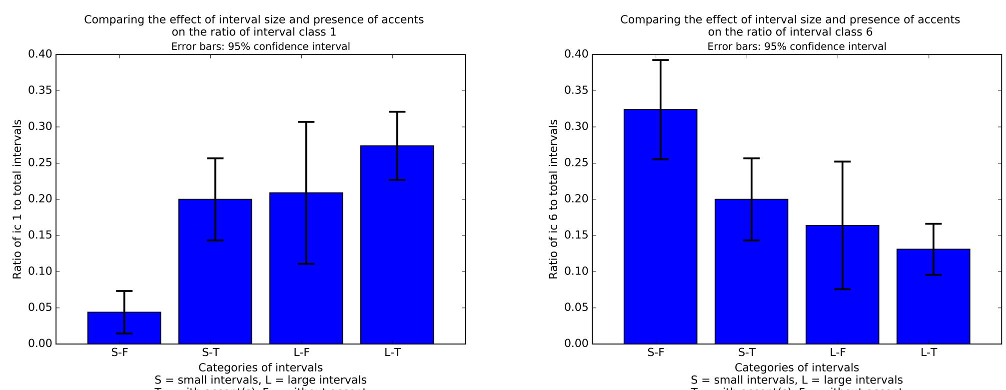


Fig. 8: Prevalence of interval classes 1 and 6 for intervals divided into categories of small or large interval size and with or without accent(s). (Small intervals are no larger than two octaves.)

The plot on the left of Figure 8 shows the ratio of interval class 1 for each category, while the plot on the right shows the ratio of interval class 6. (Recall that interval classes 1 and 6 had the highest standardized residuals in Table 1 and were most responsible for the divergence between observed and expected interval frequencies.) As the categories progress from 'small intervals without accents' to 'large intervals with accents' there is a clear trend in the ratios of both interval classes from their (de-)emphasis at the beginning of the étude toward their expected ratios corresponding to the inter-harmonies, though not all of the differences between categories are significant. Table 3 summarizes the significance and effect size for interval classes 1 and 6 when holding interval size constant while varying presence of accents and vice versa. For both interval classes there is a significant and moderate effect of the presence of accents when the interval size is small and of interval size when no accents are present. There is a small and borderline significant effect of interval size when accents are present and a notable lack of effect of the presence of accents when the interval size is large. Taken together, the prevalence of these two interval classes differs noticeably from the inter-harmonies when both the size of the interval is not greater than two octaves and no accents are present; when neither of these conditions is present, the prevalence of these interval classes does not differ strongly from the inter-harmonies. Thus, both explanations seem to be justified—the isorhythmic structure limited Ligeti's freedom in controlling note-against-note harmonies and Ligeti exerted less control over harmonic intervals as the hands drifted apart, forming independent streams, and focusing the listener's attention on the intra- rather than inter-harmonies.

categories	ic 1	ic 6
small size, accent varies	$p < .0001, \phi = .23$	$p < .01, \phi = .14$
large size, accent varies	$p = .34, \phi = .05$	$p = .60, \phi = .03$
no accent, size varies	$p < .001, \phi = .24$	$p < .02, \phi = .15$
1 or 2 accents, size varies	$p = .07, \phi = .08$	$p < .05, \phi = .09$

Table 3: Significance and effect size for prevalence of interval classes 1 and 6 depending on interval size and presence of accents

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