Sturmian Canons

Clifton Callender

Florida State University
clifton.callender@fsu.edu

Abstract. Sturmian words are balanced, almost periodic, self-similar and hierarchical infinite sequences that have been studied in music theory in connection with diatonic scale theory and related subjects. Carey and Camplitt (1996) give a brief but suggestive rhythmic example in which these properties are made manifest in a particularly visceral manner. The present paper expands upon this example, considering the properties of canons based on Sturmian words, or Sturmian canons. In particular, a Sturmian word of irrational slope a with a hierarchical periodicity of p gives rise to p-tuple canons, the voices and relations of which are determined by the terms of the continued fraction expansion of a.

Keywords: Sturmian word, canon, self-similarity

1 Introduction

Suppose a percussionist wishes to play a steady pulse divided exclusively between a high- and a low-pitched instrument. Furthermore, she wishes for the ratio of high (H) to low (L) attacks to remain constant on average and for these attacks to be distributed as evenly as possible. If the ratio is relatively simple, then it is fairly straightforward to determine the appropriate sequence of L and H. For example, if there are three high attacks for every five pulses, she must play a rotation of ||: (L H) (L H H) :||, dividing the five-beat pattern into a $2 + 3$ rhythm. In the case of eight high attacks for every 13 pulses, then the thirteen-beat pattern is divided into a $(2+3)+(2+3+3)$ rhythm, yielding the sequence ||: ((L H) (L H H)) ((L H) (L H H) (L H H)) :||.

In the case of more complicated ratios, it is helpful to think of the rhythmic pattern as the *lower mechanical sequence* of slope a, a discretization of the line $y = ax$, where the slope is the number of high attacks per some number of pulses. Given $a \in \mathbb{R}$, the lower mechanical sequence of slope a, $c_a : \mathbb{N} \to \{0, 1\}$, is given by

$$c_a(n) = \lfloor (n+1)a \rfloor - \lfloor na \rfloor - \lfloor a \rfloor \ .^1$$

(Note that it is only the fractional part of a that is important for the mechanical sequence. That is, $c_a = c_{a+j}, j \in \mathbb{Z}$.) We can define the retrograde of c_a, written $\overleftarrow{c}_a : \mathbb{N} \to \{0, 1\}$, as the *upper* mechanical sequence:

$$\overleftarrow{c}_a(n) = \lceil (n+1)a \rceil - \lceil na \rceil - \lceil a \rceil \ .$$

[1] Throughout this paper, $\mathbb{N}$ represents the natural numbers including zero, while $\mathbb{N}^+$ excludes zero.

If the slope is rational with $a = {}^{p}/_{q}$ in reduced form, then the sequence will repeat with a period of q. For example, if we have 21 high attacks for every 34 pulses, then the sequence $c_{21/34}$ is

$$(((01)\ (011))\ ((01)\ (011)\ (011)))\ (((01)\ (011))\ ((01)\ (011)\ (011))\ ((01)\ (011)\ (011)))\ ,$$

which is expressed musically in Figure 1 by associating low attacks with 0s and high attacks with 1s. Note that the sequence can be segmented into progressively higher-level groups of 2, 3, 5, 8, and 13 pulses indicated by parentheses in the binary sequence above and by the beaming and placement of bar lines in the realization. The lowest-level grouping, indicated by breaks in the sixteenth-note beam, always begins with an isolated L followed by either one or two H, yielding short and long groupings of two and three sixteenths, respectively.

Fig. 1. Realization of $c_{21/34}$

If the slope is irrational, $a \in \mathbb{R}\backslash\mathbb{Q}$, then the mechanical sequence is aperiodic. For example, consider the mechanical sequence whose slope is the reciprocal of the golden ratio, $\Phi = {}^{1}/_{\phi} = \text{frac}(\phi) \approx 0.618\ldots$, with a corresponding musical realization in Figure 2. The first 34 pulses of the aperiodic sequence (top line of Figure 2) are the same as the 34-pulse rhythmic pattern in Figure 1, since ${}^{21}/_{34}$ is a very close approximation of Φ. (In fact, the two sequences will not diverge until the 89$^{\text{th}}$ pulse.) The hierarchical groupings of the top line are made explicit in the slower moving lines below. Each of the bottom four lines are derived by placing an attack only where the line immediately above has a low note and assigning low and high attacks to short and long durations, respectively.

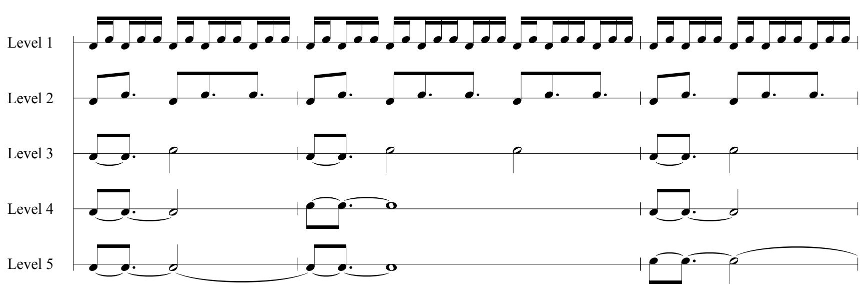


Fig. 2. Realization of c_{Φ}

For irrational slopes, the resulting mechanical sequences are Sturmian words, which have many well-known properties made manifest in the rhythmic structure of Figure 2. Sturmian words are *balanced*, as evidenced by the maximally-even distribution ([9]) of high and low attacks; *aperiodic*, corresponding to the lack of repeats; and hierarchical, as evidenced by the recursively derived sequences of low and high attacks. While Sturmian words have been studied in music theory in connection with diatonic scale theory and related subjects (see [8] and [11] among many others), Canright ([6]) as well as Carey and Clampitt ([7]) have discussed specific examples of rhythms based on Sturmian words and both serve as a point of departure for the present paper, which examines more generally the self-similar and hierarchical properties of these rhythmic structures as they manifest in Sturmian canons.

After a brief description of the hierarchy associated with each Sturmian word, this paper will focus on two issues concerning Sturmian canons. First, note that another property of Sturmian words is that they have the potential for self-similarity, which is evident in Figure 2.[2] A significant feature of the rhythmic hierarchy is that every level has the same sequence of low and high attacks as the original, revealing a high degree of self-similarity in the sequence c_{Φ}. In a sense, each level is an augmentation of the sequence of low and high attacks in the previous levels and a diminution of the sequence in subsequent levels. However, the various lines are not exact rhythmic augmentations and diminutions, because the ratio of durations does not remain constant, as is evident by comparing the pulse stream of equal sixteenth notes in the top line with the unequal durations of subsequent lines. At progressively higher levels the ratios of short to long durations converge on Φ and thus the relationship between successive lines becomes closer to exact augmentation and diminution. Indeed, the duration ratios of levels three and four are so close that the lines are perceived to be the same rhythmic pattern moving at different tempos, yielding a rhythmic tempo canon. But the perception of a *canonic* relationship between levels one and two is much weaker.

The second issue concerns the underlying structure of Sturmian canons. For example, while the canonic potential of c_{Φ} is readily apparent from the self-similar lines of Figure 2, it is not clear by inspecting the first few lines of Figure 3 that the sequence $c_{\sqrt{10}/2}$ has any canonic potential at all. The second line does not being like the first (LLH as opposed to LH), and the third and fourth lines begin on high rather than low notes. Perhaps the fifth level is the same sequence of L and H as the first, but this not clear from the limited excerpt—it is possible that level five continues with another high note, thus beginning LHH. The contrast between the apparent canonic potential of the two sequences is particularly striking given the surface similarities of the rhythmic realizations, especially on the first level. However, the lack of canonic potential evident in the first few levels of Figure 3 is misleading. In fact, $c_{\sqrt{10}/2}$ gives rise to a quadruple

---

[2] The current paper is thus connected with other recent examples of self-similar musical structures, including [1], [2], [12], and ongoing work of mine on canons with infinite solutions.

canon with consecutive voices (not levels!) related by both augmentation and retrograde.

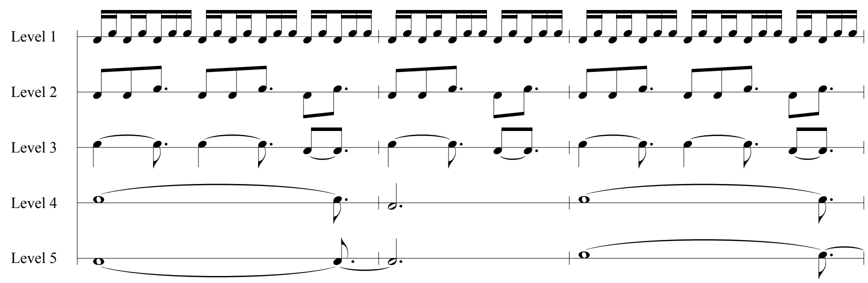


Fig. 3. Realization of $c_{\sqrt{10}/2}$

Section 4 demonstrates how to answer questions regarding the canonic structure for the rhythmic hierarchy associated with any mechanical sequence based entirely on the continued fraction expansion of the slope. These questions include:

1. Is the rhythmic hierarchy canonic?
2. If so, what is the exact augmentation/diminution factor between canonic voices?
3. Are the voices retrogrades of one another?
4. How many different canons are manifest in a single rhythmic hierarchy?

The answer to the last question opens up the possibility of double, triple, and n-tuple Sturmian canons.[3]

2 Hierarchical structure

As suggested by the examples above, any Sturmian word gives rise naturally to a hierachy of infinite levels with the original sequence being level one, which is then segmented into groups, or *runs*, on level two, runs of runs on level three, and so on. In this section we give a relatively informal description of the run heirarchies of Sturmian words borrowing heavily from [13], to which the reader is referred for the formal details.

For any level $k \geq 2$ runs come in either a short, S_k, or a long, L_k, form, with long runs containing exactly one more run from the previous level than short runs, and with $S_1 = 0$ and $L_1 = 1$. On each level, either short or long runs will

[3] Audio relatizations of selected examples in this paper and a Max/MSP patch for generating novel canons based on any real number are available at `cliftoncallender.com`.

be more frequent. If short runs are more frequent, there will be strings of one or more consecutive S_k separating singly occurring L_k: $\dots S_k^m L_k S_k^m \dots$; if long runs are more frequent, then the opposite case holds. In addition, each level will begin with either short or long runs, depending on the previous level's sequence of runs. Thus, on each level k, runs are grouped into one of four different forms to become a run on level $k+1$: $S_k^m L_k$, $L_k S_k^m$, $S_k L_k^m$, and $L_k^m S_k$, where m is one of two consecutive natural numbers, depending on whether the prevailing run on level $k+1$ is short or long.

For c_a, the specific form for each level k can be determined directly from the continued fraction expansion of a. Beginning with the specific case in which the fractional part of the continued fraction contains no term equal to one ($a = [a_0; a_1, a_2, \dots], a_{n\in\mathbb{N}^+} \neq 1$) and keeping this assumption in mind, the run forms occur as follows:

1. For level k, the relevant term of the continued fraction is a_k.
2. Since the assumption is that $a_k \neq 1$, short runs are more frequent.
3. The number of short runs separating each long run is either a_k or $a_k - 1$, depending on whether the prevailing run on level $k+1$ is long or short. That is, $S_{k+1} = S_k^{a_k-1} L_k$ or $L_k S_k^{a_k-1}$, $L_{k+1} = S_k^{a_k} L_k$ or $L_k S_k^{a_k}$.
4. Level k begins with S_k or L_k depending on whether k is odd or even, respectively.

For continued fractions with terms that are equal to one, the situation is slightly more complicated. If the relevant term for level k is $a_j = 1$, then, first, long runs will be more frequent with one or more L_k separating singly occurring S_k. Second, the number of long runs occurring consecutively will be either $a_j + a_{j+1}$ or $a_j + a_{j+1} - 1$, depending on whether the prevailing run on level k is long or short. Third, the relevant term for level $k+1$ is not a_{j+1}, as might be expected, but a_{j+2}. This is necessary because of the unique role that continued fraction terms of "1" play in the run hierarchy; terms that are preceded by an odd number of consecutive "1"s are skipped over by what is defined in [13] as the *index jump function* (adapted slightly to fit the present purposes):

Definition 1. ***(Uscka-Wehlou 2009)*** *For each $a \in \mathbb{R} \setminus \mathbb{Q}$ the index jump function $i_a : \mathbb{N}^+ \to \mathbb{N}^+$ is defined by $i_a(1) = 1$ and $i_a(k+1) = i_a(k)+1+\delta_1(a_{i_a(k)})$ for $k \geq 1$, where $\delta_j(x)$ equals 1 if $x = j$ and 0 otherwise and $a_1, a_2, \dots \in \mathbb{N}^+$ are the continued fraction terms of a.*

Putting the foregoing together we have:

1. The relevant term for level k is $a_{i_a(k)}$.
2. If $a_{i_a(k)} = 1$, then long runs are more frequent; otherwise, short runs are more frequent.
3. More frequent runs, either long or short, occur in string of either m_k or $m_k - 1$ consecutive runs, where $m_k = 1 + a_{i_a(k)+1}$ if $a_{i_a(k)} = 1$, and $m_k = a_{i_a(k)}$ otherwise.
4. Level k begins with either S_k or L_k, depending on whether $i_a(k)$ is odd or even, respectively.

3 Run-length and duration ratios

Now that we have a description of the run hierarchy we turn to measuring the length and duration of runs in order to construct true rhythmic canons and to understand their structure. First, we let $|S_k|$ and $|L_k|$ represent the (binary-word) length of short and long runs on level k measured in terms of the length of the corresponding word in the original sequence:

Theorem 1. ***(Uscka-Wehlou 2009)*** *Let $a \in \mathbb{R} \setminus \mathbb{Q}$ and $a = [a_0; a_1; a_2, \dots]$. For c_a we have for all $k \in \mathbb{N}^+$:*

$$|S_k| = q_{i_a(k+1)-1} \quad \text{and} \quad |L_k| = q_{i_a(k+1)-1} + q_{i_a(k+1)-2},$$

where i_a is the index jump function, $|S_k|$ and $|L_k|$ for $k \in \mathbb{N}^+$ denote the (binary word)-length of short, respectively long runs of level k, and q_k are the denominators of the convergents on the continued fraction expansion of a.

For example, given $a = \sqrt{10}/2 = [a; 1, 1, 2, 1, 1, 2, \dots]$, we have

k	1	2	3	4	5
$\|S_k\|$	1	2	5	12	31
$\|L_k\|$	1	3	7	19	43

which corresponds precisely to the lengths (measured in sixteenth notes) of low and high notes at each level in Figure 3.

As a consequence of the theorem, we can calculate ratios for the lengths of long and short runs, the run-length ratio, for any level of the hierarchy, using the following original corollary:

Corollary 1. *Let $a \in \mathbb{R} \setminus \mathbb{Q}$ and $a = [a_0; a_1, a_2, \dots]$. For c_a the run-length ratio of level k is 1 if $k = 1$ and*

$$\frac{|L_k|}{|S_k|} = [1; a_{i_a(k-1)}, a_{i_a(k-2)}, \dots, a_{i_a(1)} = a_1]$$

if $k > 1$.

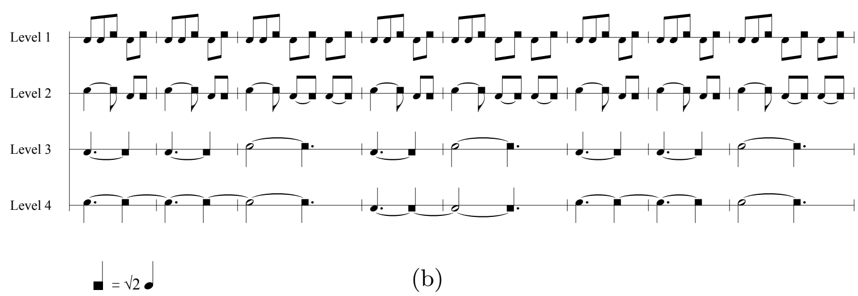
Example: $a = \sqrt{2} = [1; \overline{2}]$

According to the corollary, the sequence of ratios $|L_k|/|S_k|$ is $|L_1|/|S_1| = [1; 0] = 1/1$, $|L_2|/|S_2| = [1; 2] = 3/2$, $|L_3|/|S_3| = [1; 2, 2] = 7/5$, with the series converging to $\lim_{k\to\infty} |L_k|/|S_k| = [1; \overline{2}]$. The fact that the run-length ratios converge to a *single* value indicates that the sequence of runs for *every* level of the hierarchy is either identical to or the retrograde of the original sequence, $c_{\sqrt{2}}$. This property is true for only special cases of c_a where a is either $[a_0; \overline{n}]$ or $[a_0; \overline{1, n}], n \in \mathbb{N}^+$.

However, as a *rhythmic* sequence, this is not the case, since the ratios of *durations* are not constant. Note that in Figure 4a no level is related to any other level by exact rhythmic augmentation. We can rectify the situation in the following manner. Let $||S_k||$ and $||L_k||$ be the *rhythmic duration* of S_k and L_k, respectively, and let $\beta_1 = [1; \overline{2}]$. By using irrational duration ratios, such that

$||L_1|| = \beta_1 \cdot ||S_k||$, we can ensure that the duration ratio between long and short runs is constant at every level of the hierarchy, $||L_k||/||S_k|| = \beta_1$, $k \in \mathbb{N}^+$. See Figure 4b, in which durations with square note heads are $\sqrt{2}$ times as long as their oval counterparts.[4]

(a)



(b)

Fig. 4. (a) Realization of the first four levels of $c_{\sqrt{2}}$. In each level, the lower note corresponds to S_k and the upper note to L_k. Note that the run-duration ratios are not constant from one level to another. (b) Realization of the first four levels of $c_{\sqrt{2}}$ with irrational durations indicated by square note heads. The ratio between long and short durations for each level is $\sqrt{2}$. The durations for each level (measured in eighth notes) are $||S_1|| = 1$, $||L_1|| = \sqrt{2}$; $||S_2|| = 1 + \sqrt{2}$, $||L_2|| = 2 + \sqrt{2}$; $||S_3|| = 3 + 2\sqrt{2}$, $||L_3|| = 4 + 3\sqrt{2}$; and $||S_4|| = 7 + 5\sqrt{2}$, $||L_4|| = 10 + 7\sqrt{2}$.

[4] While difficult, it is possible to perform such rhythms either using rational approximations (see [5]) or with the aid of click tracks, and in any case they can be realized easily enough on a computer. These rhythms will tend to be perceived in terms of simpler rational ratios. For instance, I hear the rhythms of figure 4b as variants of $2 + 3$ and $2 + 2 + 3$ rhythms in which the final duration is very subtly shortened.

More generally, after defining the durations for S_1 and L_1, we can define the run-duration ratios recursively, such that

$$\begin{aligned}||S_{k+1}|| = &\left(1 + \delta_0\left(a_{i_a(k+1)}\right)\left(a_{i_a(k)} - 2\right)\right) \cdot ||S_k|| + \\ &\left(1 + \delta_1\left(a_{i_a(k+1)}\right)\left(a_{i_a(k)} - 2\right)\right) \cdot ||L_k|| \ ; \\ ||L_{k+1}|| = &\left(1 + \delta_0\left(a_{i_a(k+1)}\right)\left(a_{i_a(k)} - 1\right)\right) \cdot ||S_k|| + \\ &\left(1 + \delta_1\left(a_{i_a(k+1)}\right)\left(a_{i_a(k)} - 1\right)\right) \cdot ||L_k|| \ .\end{aligned}$$

Example: $a = \sqrt{6} = [2; \overline{2, 4}]$
Because the repeating portion of this continued fraction has a period of two, the run-length ratios will oscillate between two different converging series as k increases:

$$\begin{aligned}|L_1|/|S_1| = [1; 0], |L_3|/|S_3| = [1; 4, 2], \ldots, \\ \beta_1 = \lim_{j\to\infty} \frac{|L_{2j+1}|}{|S_{2j+1}|} = [1; \overline{4, 2}] = \frac{\sqrt{6}}{2} \ , \\ |L_2|/|S_2| = [1; 2], |L_4|/|S_4| = [1; 2, 4, 2], \ldots, \\ \beta_2 = \lim_{j\to\infty} \frac{|L_{2j+2}|}{|S_{2j}|} = [1; \overline{2, 4}] = \sqrt{6} - 1 \ ,\end{aligned}$$

where $j \in \mathbb{N}$. In this case, setting $||L_1|| = \beta_1 \cdot ||S_1||$ ensures that $||L_k||/||S_k||$ is β_1 if k is odd and β_2 if k is even. Thus, any two levels of the hierarchy that are either both odd or both even will be related by exact rhythmic augmentation/diminution. (See Figure 5.)

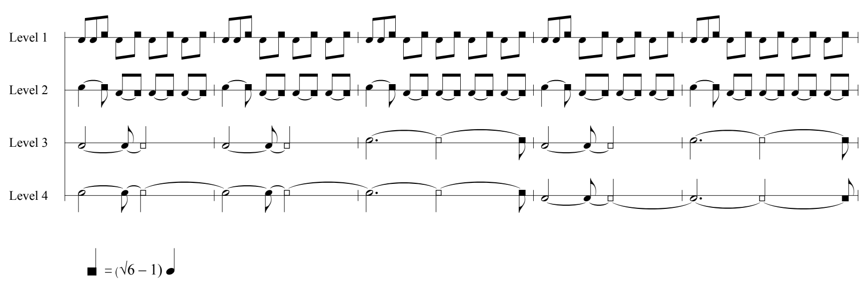


Fig. 5. Realization of the first four levels of $c_{\sqrt{6}}$.

Example: $a = [a_0; \overline{1, 1, 2, 3}]$
In this case, although the repeating portion of the continued fraction has a period of four, there is an inessential term (the second "1" in the group of four which is skipped by the index jump function), so the ratio $|L_k|/|S_k|$ will converge on three values:

$$\beta_1 = \lim_{j\to\infty} \frac{|L_{3j+1}|}{|S_{3j+1}|} = [1;\overline{3,2,1,1}] \ ,$$
$$\beta_2 = \lim_{j\to\infty} \frac{|L_{3j+2}|}{|S_{3j+2}|} = [1;\overline{1,1,3,2}] \ ,$$
$$\beta_3 = \lim_{j\to\infty} \frac{|L_{3j+3}|}{|S_{3j}|} = [1;\overline{2,1,1,3}] \ .$$

(See Figure 6.)

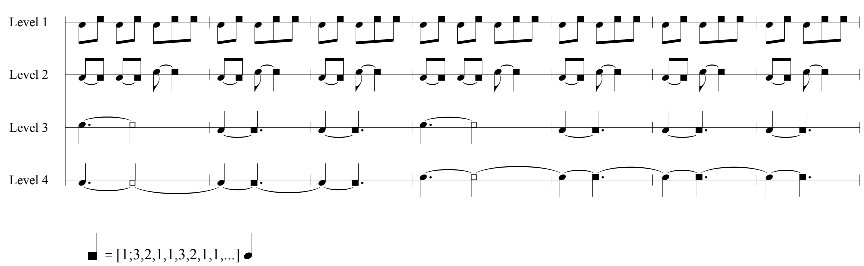


Fig. 6. Realization of the first four levels of $c_{[a_0;\overline{1,1,2,3}]}$.

More generally, for $a = [a_0; \overline{a_1, a_2, \ldots, a_n}]$, the series $||L_k||/||S_k||$ will converge to $p \leq n$ different values, $\beta_1, \ldots, \beta_p$. (Specifically, noting that the ratio of k and $i_a(k)$ converges, $p = n \lim_{k\to\infty} {}^{k}/_{i_a(k)}$.) We will say that p is the *hierarchical periodicity* of c_a. In all cases, setting $||L_1|| = \beta_1 \cdot ||S_1||$ will ensure that the run-duration ratio is constant for all levels:

$$\frac{||L_k||}{||S_k||} = \beta_{k \bmod p} \ .$$

(We will refer to β_0 as β_p.)

4 Structural properties of Sturmian p-tuple canons

In this section, we describe the canonic structure arising from the hierarchical levels of c_a when a has a repeating continued fraction or, equivalently, a is a quadratic surd.

Let $a = [a_0; \overline{a_1, a_2, \ldots, a_n}]$ and p be the hierarchical periodicity of c_a with run-length convergents of $\beta_1, \ldots, \beta_p$. If $||L_1||/||S_1|| = \beta_1$, then the resulting hierarchical rhythmic structure is a Sturmian p-tuple canon, written C_a.

The p individual canons of C_a, canon 1, canon 2, and so on to canon p, are written $C_{a,1}, C_{a,2}, \ldots, C_{a,p}$. For $1 \leq j \leq p$, the hierarchical levels $k = j + p, j + 2p, \ldots$ constitute the consecutive voices of canon j. If p and n are opposite in parity (one being odd and the other being even), then consecutive voices of each canon, 1 through p, are retrogrades of one another. For all p, odd or even, any voice of a given canon is an exact rhythmic augmentation of the previous voice by a factor of $||S_{p+1}||$. That is, for any S_k and L_k, $||S_{k+p}|| = ||S_k|| \cdot ||S_{p+1}||$ and $||L_{k+p}|| = ||L_k|| \cdot ||S_{p+1}||$.

Given the cyclic nature of the p-tuple canon, identifying any one of the individual canons with level 1 is arbitrary. In reality, the p individual canons form a family of canons that always occur together. Beginning with any one of the individual canons will lead to the exact same p-tuple structure.

As an application of the above, we detail the canonic structure of the three examples from the Section 3.

Example: $a = \sqrt{2} = [a_0; \overline{2}]$

Since the hierarchical period of c_a is one, the corresponding Sturmian canon consists of a single canonic rhythm imitated in augmentation and possibly retrograde at all levels of the hierarchy. Specifically, consecutive voices are related by retrograde and augmentation by a factor of $||S_2|| = 1 + \sqrt{2}$. We can graph the canonic structure of $C_{\sqrt{2}}$ as in Figure 7, with double arrows indicating retrograde and with both the augmentation factor and run-duration ratios included.

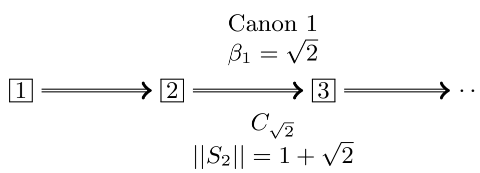


Fig. 7. Graph of canonic structure for $C_{\sqrt{2}}$. The numbers $\boxed{1}$,$\boxed{2}$,... represent successive levels of the hierarchy. Double arrows indicate that the levels are in a retrograde relationship. $||S_2||$ gives the augmentation factor between consecutive voices of the canon and β_1 gives the ratio between long and short duration at each level.

Example: $a = \sqrt{6} = [2; \overline{2, 4}]$

Since the hierarchical period of $c_{\sqrt{6}}$ is two, $C_{\sqrt{6}}$ is a Sturmian *double* canon with two different rhythmic patterns occurring at alternate levels. Consecutive voices of each canon occur at every other level of the hierarchy, are not related by retrograde (since both p, the hierarchical period, and n, the period of the repeating portion of the continued fraction, are even) and are augmentations by a factor of $||S_3|| = 5 + 4 \cdot [1; \overline{4, 2}] = [9; \overline{1, 8}]$. The corresponding graph of the canonic structure of $C_{\sqrt{6}}$ is given in Figure 8.

It is interesting to note that while the fractional parts of $\sqrt{2}$ and $\sqrt{6}$ are very close, differing by only $0.035\ldots$, their respective canonic structures are very different. Indeed, the structure of $C_{\sqrt{2}}$ is much more similar to C_ϕ, both being single canons with runs on each level k being comprised of two or three runs from level $k-1$. The main differences between $C_{\sqrt{2}}$ and C_ϕ is that the former is dominated by short runs and consecutive voices are related by retrograde while the latter is dominated by long runs and contains no retrogrades. Both of these properties for the latter are a consequence of the continued fraction expansion of $\phi = [1; \overline{1}]$, in which all terms are equal to 1.

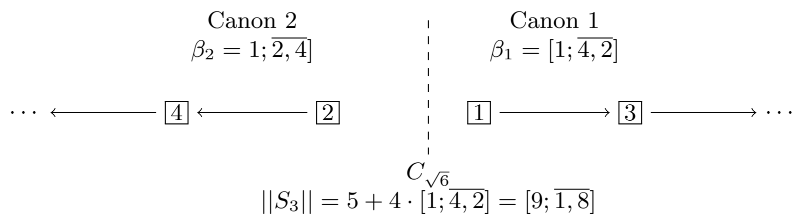


Fig. 8. Graph of canonic structure for $C_{\sqrt{6}}$.

Example: $a = [a_0; \overline{1,1,2,3}]$

As noted before, the hierarchical period of $c_{[a_0;\overline{1,1,2,3}]}$ is three rather than four due to the presence of an inessential "1" in the terms of the repeated continued fraction. Thus, $C_{[a_0;\overline{1,1,2,3}]}$ is a Sturmian *triple* canon with three different rhythmic patterns occurring in a cycle of length three. Consecutive voices of each canon occur at every third level of the hierarchy, are related by retrograde (since p is odd while n is even) as well as augmentation by a factor of $||S_4|| = 7 + 10 \cdot [1; \overline{3}, 2, 1, 1] = [19; \overline{1, 18}]$. The corresponding graph of the canonic structure is given in Figure 9.

5 Suggestions for further research

We conclude with a couple of suggestions for further research or compositional application. Similar sequences with more than two elements arise from the discretization of lines in more than two dimensions [4]. For example, suppose we want a rhythmic sequence of three elements in the ratios $a : b : c$ and with the attacks of each element distributed as evenly as possible. The resulting sequence is a discretization of the line $\langle x, y, z \rangle = \langle at, bt, ct \rangle$. Figure 10 is a musical realization of the beginning of a sequence with low-, middle-, and high-pitched instruments in the ratios $1 : \phi : e$. More generally, we can consider rhythmic sequences based on continuous curves not limited to straight lines. For example, we can create a sequence that gradually morphs from one Sturmian word to another, c_{α_1} and c_{α_2}, by allowing the slope to vary gradually from α_1 to α_2.

Sturmian canons can also be used as the basis for tiling canons. The typical rhythmic tiling, as discussed in [3], [10], and [14], is periodic with tiles of finite length. For tilings based on a Sturmian canon both tiles and tiling are aperiodic

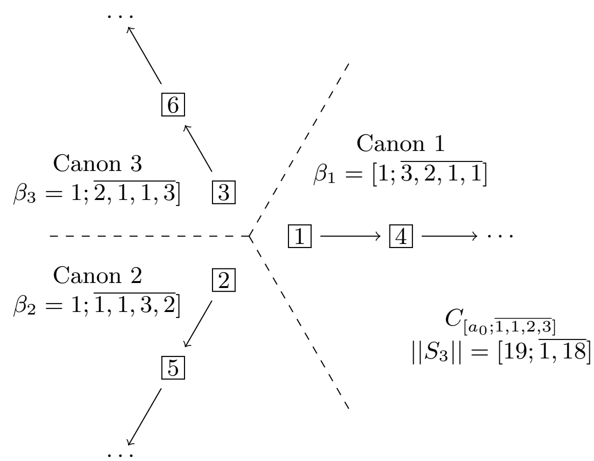


Fig. 9. Graph of canonic structure for $C_{[a_0; \overline{1,1,2,3}]}$.

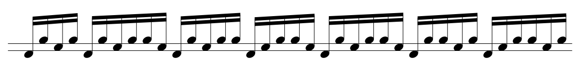


Fig. 10. Rhythmic sequence based on a maximally-even distribution of three instruments with attcks in the ratio $1 : \phi : e$.

(specifically, almost periodic) and the composite rhythm is often a diminution of one or more tiles. Figure 11 presents a simple tiling based on $c_{\sqrt{2}}$, where voice one is level two, voice two is level two with a slight delay, and voice three is level three with a slightly longer delay. The three voices are all exact augmentations (and retrogrades, in the case of voices one and two) of the composite rhythm.

References

1. Adler, C.: Signals Intelligence, for percussion solo or ensemble. Liber Pulveris Press (2002)
2. Amiot, E.: Auto Similar Melodies. Journal of Mathematics and Music 3.1, 1–26 (2009)
3. Amiot, E., Rahn, J. (eds.): Tiling Rhythmic Canons [Special Issue]. Perspectives of New Music 49.2 (2011)
4. Berthé, V., Labbé, S.: Uniformly balanced words with linear complexity and prescribed letter frequencies. In: Electronic Proceedings in Theoretical Computer Science, pp. 44–52. (2011)
5. Callender, C.: Performing the Irrational: Paul Ushers Arrangement of Nancarrows Study no. 33, Canon $2 : \sqrt{2}$. Online Symposium: Conlon Nancarrow, Life and Music (2012) `http://conlonnancarrow.org/symposium/papers/callender/irrational.html`
6. Canright, D.: Fibonacci Gamelan Rhythms. 1/1: The Journal of the Just Intonation Network 6.4 (1990)

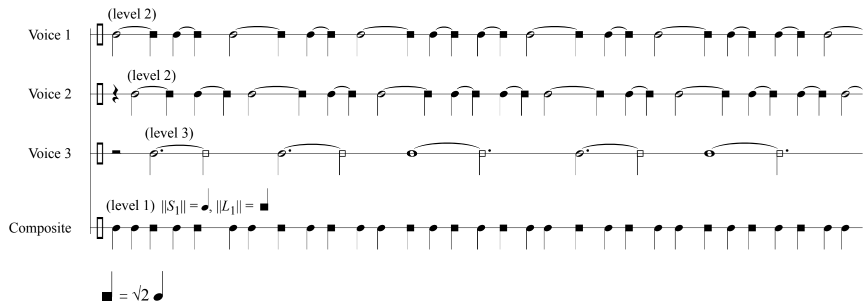


Fig. 11. Three-voice tiling of $c_{\sqrt{2}}$ with tiles drawn from levels two and three.

7. Carey, N., Clampitt, D.: Self-Similar Pitch Structures, Their Duals, and Rhytymic Analogues. Perspectives of New Music 34, 62–87 (1996)
8. Clampitt, D., Domínguez, M., Noll, T.: Plain and Twisted Adjoints of Well-Formed Words. In: Chew, E., Childs, A., Chuan, C.H. (eds.) Proceedings of the Second International Conference on Mathematics and Computation in Music, pp. 65–80. Springer, Berlin (2009)
9. Clough, J., Douthett, J.: Maximally Even Sets. Journal of Music Theory 35, 93–173 (1991)
10. Hall, R., Klinsberg, P.: Asymmetric Rhythms and Tiling Canons. The Mathematical Association of America 113, 887–896 (2006)
11. Noll, T.: Sturmian Sequences and Morphisms: A Music-Theoretical Application. In: Yves André: Mathématique et Musique. Journée Annuelle de la Société Mathématique de France à lInstitut Henri Poincaré (2008)
12. Shallit, J.: Mathematics of Per Nøgård's Rhythmic Infinity System. The Fibonacci Quarterly 27.2, 262–268 (2005)
13. Uscka-Wehlou, H.: Run-hierarchical structure of digital lines with irrational slopes in terms of continued fraction and the Gauss map. Pattern Recognition 42, 2247–2254 (2009)
14. Vuza, D.: Supplementary Sets and Regular Complementary Unending Canons. Perspectives of New Music (Part One) 29.2, 22–49, (Part Two) 30.1, 184–207, (Part Three) 30.2, 102–125, (Part Four) 31.1, 207–305 (1991–1993)